Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 1, pp. 140–146, 2002

# STRESSES IN A CONICAL TUBE UNDER SUDDEN LOADING

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UDC 539.3

*The paper considers a thick-walled long conical tube from an ideal plastic material whose inner surface is suddenly subjected to time-constant, uniformly distributed pressure or is given a velocity. An ideal-plastic zone propagates from the inner conical surface. It is assumed that the material of the tube is incompressible in both the elastic and plastic zones. The plastic material obeys the Houber–Mises plasticity condition.*

Rakhmatulin and Dem'yanov [1] and Agababyan [2] considered the elastoplastic problem for a cylindrical tube. Ivlev [3] treated the static problem of the limiting state of a conical tube, and Sokolovskii [4] studied the elastoplastic state of the tube.

**1. Motion of a Tube under Suddenly Applied Internal Pressure.** We consider the limiting state of a cylindrical tube from an ideally plastic incompressible material which is suddenly subjected to time-constant internal pressure. The corresponding elastoplastic problem reduces to the differential equation [2]

$$Cx'' + \ln x - x/\delta - p/k + 1 = 0, \tag{1}$$

where $C = a^2\rho \ln\delta/(2G)$, $\delta = b/a$, $x = r_*^2(t)/a^2$, $r = r_*(t)$ is the equation of the cylindrical interface between the elastic and plastic zones, and $a$ and $b$ are the radii of the inner and outer cylindrical surfaces, respectively. The initial conditions are written as

$$x(t_0) = 1, \qquad x'(t_0) = \sqrt{2/(Ck)}\sqrt{p - p_0}, \qquad p_0 = (k/2)(1 - a^2/b^2).$$

Here $p_0$ is the minimum value of $p$ for which plastic strains occur on the surface $r = a$ and $t_0$ is the moment the plastic strains start to propagate.

Introducing the new function $x' = \Phi(x)$, we reduce Eq. (1) to the first-order differential equation

$$C(x'^2(t) - x'^2(t_0)) + 2x\ln x - (x-1)(2p/k + (x+1)/\delta^2) = 0. \tag{2}$$

We determine the limiting pressure $p = p_*$ at which the tube becomes entirely plastic at $t = t_*$. Setting $x'(t_*) = 0$ and $x(t_*) = \delta^2$ in (2), we obtain

$$p_* = 2k\ln(b/a) - (k/2)(1 - a^2/b^2). \tag{3}$$

Here the first term is the limiting plastic pressure and the second term is half the elastic pressure in the static problem [5].

Let pressure $p(t) = \text{const}$ be applied on the inner surface $\theta = \alpha$ of the thick-walled tube at the moment $t = 0$ (see Fig. 1). The condition of axial symmetry implies that $w = 0$. In solving a similar elastoplastic problem, Sokolovskii [4] assumed that $u = \tau_{r\theta} = 0$. In [1, 2], these components are absent in a similar dynamic problem for a cylindrical tube. Apparently, this assumption is valid only for a slightly conical tube. As a result, the equation of motion becomes

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\sigma_\theta - \sigma_\varphi}{r}\cot\theta = \rho\frac{\partial^2 v}{\partial t^2}, \qquad \sigma_r = \frac{\sigma_\theta + \sigma_\varphi}{2}.$$

The strain components are related to displacements by the formulas $\varepsilon_\theta = (1/r)(\partial v/\partial\theta)$ and $\varepsilon_\varphi = (v/r)\cot\theta$.

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Institute of Mechanics, Armenian National Academy of Sciences, Erevan 375019. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 43, No. 1, pp. 168–176, January–February, 2002. Original article submitted December 8, 2000; revision submitted July 13, 2001.

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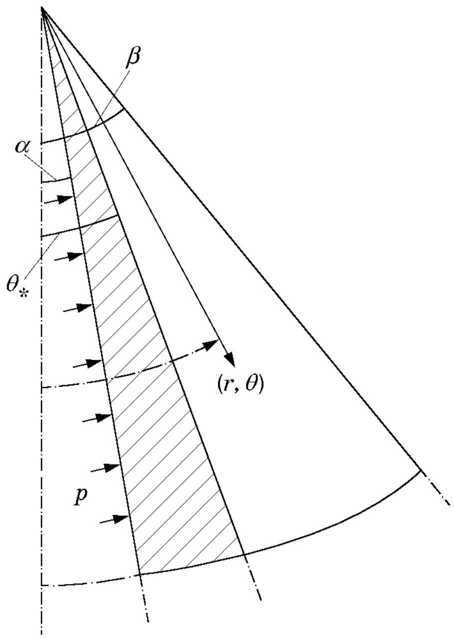


Fig. 1

The incompressibility condition $\varepsilon_\theta + \varepsilon_\varphi = 0$ implies that $v = \psi(r,t)/\sin\theta$, where $\psi(r,t)$ is an arbitrary function of $r$ and $t$.

1.1. *Linearly Elastic State.* When the applied pressure $p$ is low, the tube is in the purely elastic state. Using Hooke's law, we obtain

$$\sigma_\varphi - \sigma_\theta = \frac{4G}{r}\,\frac{\cos\theta}{\sin^2\theta}\,\psi(r,t). \tag{4}$$

Substitution of Eq. (4) into the equation of motion yields

$$\frac{\partial\sigma_\theta}{\partial\theta} = \frac{4G}{r}\,\frac{\cos\theta}{\sin^2\theta}\,\psi(r,t) + \frac{\rho r}{\sin\theta}\,\frac{\partial^2\psi}{\partial t^2}.$$

Integrating this equation and using the boundary condition on the outer conical surface ($\sigma_\theta = 0$ for $\theta = \beta$), we obtain

$$\sigma_\theta = \frac{2G}{r}\,\psi(r,t)\Big(\frac{\cos\beta}{\sin^2\beta} - \frac{\cos\theta}{\sin^2\theta} + \ln\frac{\tan\,(\beta/2)}{\tan\,(\theta/2)}\Big) - \rho r\,\frac{\partial^2\psi}{\partial t^2}\,\ln\frac{\tan\,(\beta/2)}{\tan\,(\theta/2)}. \tag{5}$$

Using (5), with allowance for the boundary condition at the inner surface ($\sigma_\theta = -p$ for $\theta = \alpha$), we find that

$$r\,\frac{\partial^2\psi}{\partial t^2} + \frac{\omega^2}{r}\,\psi(r,t) = Q. \tag{6}$$

Here

$$\omega^2 = 2G\omega_0^2(\alpha)/(\rho g(\alpha)), \qquad Q = p/(\rho g(\alpha)), \tag{7}$$

and $\omega_0^2(\theta) = \cos\theta/\sin^2\theta - \cos\beta/\sin^2\beta - g(\theta)$, where $g(\theta) = \ln\,(\tan\,(\beta/2)/\tan\,(\theta/2))$.

Introducing the new function $\psi(r,t) = rF(\xi)$, where $\xi = t/r$ and $0 \leqslant \xi < \infty$, we express the shear strain as $2\gamma_{r\theta} = \partial v/\partial r - v/r = -\xi F'(\xi)/\sin\theta$. For $\xi = \xi_1$, the condition that the shear stress vanishes

$$F'(\xi_1) = 0 \tag{8}$$

allows one to obtain an exact solution of the problem in the indicated neighborhood of the value of $\xi_1$. If $\tau_{r\theta} \neq 0$, the solution for the values of $\xi$ different from $\xi_1$ can be considered approximate. From (6), we have the equation

$$F'' + \omega^2 F = Q, \tag{9}$$

whose solution satisfying the homogeneous initial conditions $F(0) = F'(0) = 0$ has the form

$$F(\xi) = Q(1 - \cos\omega\xi)/\omega^2. \tag{10}$$

The stress components are calculated from the formula

$$\sigma_\theta = -2G\omega_0^2(\theta)F(\xi) - \rho g(\theta)F''(\xi). \tag{11}$$

From (4) and (9)–(11), we obtain

$$\sigma_{\theta,\varphi} = -p\,\frac{g(\theta)}{g(\alpha)} + p\Big[\frac{g(\theta)}{g(\alpha)} \mp \frac{1}{\omega_0^2(\alpha)}\Big(\frac{\cos\theta}{\sin^2\theta} \mp \frac{\cos\beta}{\sin^2\beta} \mp \ln\frac{\tan\,(\beta/2)}{\tan\,(\theta/2)}\Big)\Big](1-\cos\omega\xi). \tag{12}$$

In (12), the upper sign (minus) corresponds to $\sigma_\theta$ and the lower sign (plus) to $\sigma_\varphi$.

Displacements (2) are calculated from the formula

$$v = pr(1-\cos\omega\xi)/(2G\omega_0^2(\alpha)\sin\theta). \tag{13}$$

Thus, under the action of suddenly applied, time-constant, internal pressure $p$, the conical tube performs oscillations in the neighborhood of the state of static equilibrium with frequency $\omega/r$ variable along the tube.

We assume that at the moment $t = t_0$, plastic strains occur at the inner surface of the shell. It follows from (10) and (12) that

$$\sin^2(\omega\xi_0/2) = (k/(2p))(\sin^2\alpha/\cos\alpha)\omega_0^2(\alpha). \tag{14}$$

Hence, for the limiting case of elastic motion, where plastic strains occur on the inner surface $\theta = \alpha$, we have $\xi_0 = \pi/\omega$. From (14), we obtain

$$p_0 = \frac{k}{2}\Big(1 - \frac{\sin^2\alpha}{\sin^2\beta}\,\frac{\cos\beta}{\cos\alpha} - \frac{\sin^2\alpha}{\cos\alpha}\,\ln\frac{\tan\,(\beta/2)}{\tan\,(\alpha/2)}\Big). \tag{15}$$

We note that $p_0$ is equal to half the internal pressure in the static problem [4].

Setting $p = p_0$ in (14), we have

$$t_0 = \pi r\sqrt{\rho g(\alpha)/(2G)}/\omega_0(\alpha).$$

By virtue of (15), we obtain $\tau_{r\theta} = 0$ for $\xi = \xi_0$.

Expressions for the stresses $\sigma_x$, $\sigma_\varphi$, and $\sigma_z$ and the radial displacement $v$ in a cylindrical tube are obtained by passing to the limit $\theta \to 0$, $r \to \infty$ in (11)–(13) and (15) with $r\theta = \text{const}$:

$$\sigma_{x,\varphi} = -p\,\frac{\ln\,(b/x)}{\ln\,(b/a)} + p\Big(\frac{\ln\,(b/x)}{\ln\,(b/a)} \mp \frac{b^2-x^2}{b^2-a^2}\,\frac{a^2}{x^2}\Big)(1-\cos\gamma t), \tag{16}$$

$$u = \frac{pa^2b^2}{2G(b^2-a^2)x}(1-\cos\gamma t), \qquad p_0 = \frac{k}{2}\Big(1-\frac{a^2}{b^2}\Big).$$

Here the upper sign corresponds to $\sigma_x$ and the lower sign to $\sigma_\varphi$; $\gamma^2 = 2G(b^2-a^2)/(\rho a^2b^2\ln\,(b/a))$.

Expressions (16) coincide with the formulas obtained in [2] for a cylindrical tube. In this case, the value of $p_0$ is equal to half the internal pressure in the static problem [5].

1.2. *Elastoplastic State.* For $t > t_0$, a plastic zone propagates in the conical tube. Bearing in mind that the displacement function satisfies the incompressibility condition and the continuity condition for displacements at the interface between the elastic and plastic zones, we write the displacement function for both zones in the form

$$v = \psi(r,t)/\sin\theta.$$

The condition

$$\sigma_\varphi - \sigma_\theta = 2k, \qquad \alpha \leqslant \theta \leqslant \theta_*(r,t) \tag{17}$$

is satisfied in the plastic zone, whereas the condition

$$(2G/r)(\cos\theta_*/\sin^2\theta_*)\psi(r,t) = k \tag{18}$$

holds at the interface between the elastic and plastic zones. Here $\psi(r,t)$ is a new unknown function of $r$ and $t$ and $\theta = \theta_*(r,t)$ is the interface between the plastic and elastic zones.

Introducing the function $F(\xi) = \psi(r,t)/r$, from (18) we obtain

$$\cos\theta_* = \sqrt{1+\mu^2F^2} - \mu F, \qquad \mu = G/k. \tag{19}$$

Similarly to (5), for the elastic zone, we have

$$\sigma_\theta = -2G\omega_0^2(\theta)F(\xi) - \rho g(\theta)F''(\xi) \qquad (\theta_* \leqslant \theta \leqslant \beta) \tag{20}$$

and for the plastic zone,

$$\frac{\partial \sigma_\theta}{\partial \theta} = 2k \cot\,\theta + \rho\,\frac{r}{\sin\theta}\,\frac{\partial^2 \psi}{\partial t^2}.$$

Integrating this equation and using the boundary condition at the inner surface, we obtain

$$\sigma_\theta = -p + 2k \ln\frac{\sin\theta}{\sin\alpha} + \rho \ln\frac{\tan\,(\theta/2)}{\tan\,(\alpha/2)}\,F''(\xi). \tag{21}$$

Satisfying the continuity condition for stresses $\sigma_\theta$ on the surface $\theta = \theta_*(\xi)$, we arrive at the equation

$$F'' + \nu\Omega(F) = \nu p/(2k), \qquad \nu = 2k/(\rho g(\alpha)), \tag{22}$$

where

$$\Omega(F) = \ln\frac{\sin\theta_*(F)}{\sin\alpha} + \mu F\Big(\frac{\cos\theta_*(F)}{\sin^2\theta_*(F)} - \frac{\cos\beta}{\sin^2\beta} - \ln\frac{\tan\,(\beta/2)}{\tan\,(\theta_*(F)/2)}\Big).$$

With allowance for (19), we obtain

$$\begin{aligned}\Omega(F) = 1/2 - \mu F\cos\beta/\sin^2\beta + \ln\big(\sqrt{2\mu F}\;(\sqrt{1+\mu^2F^2} - \mu F)^{1/2}/\sin\alpha\big)\\ - \mu F \ln\big(\tan\,(\beta/2)(1 - \mu F + \sqrt{1+\mu^2F^2})(\sqrt{1+\mu^2F^2} + \mu F)^{1/2}/(2\mu F)^{1/2}\big).\end{aligned} \tag{23}$$

Introducing the function $\phi$,

$$\frac{dF}{d\xi} = \phi(F), \tag{24}$$

we reduce (22) to the first-order equation

$$\phi'\phi + \nu\Omega(F) = \nu p/(2k). \tag{25}$$

From this equation, we obtain

$$\phi^2(F) = \phi^2(F_0) + 2\nu \int\limits_{F_0}^{F} \Big(\frac{p}{2k} - \Omega(x)\Big)\,dx, \tag{26}$$

where $F_0$ is the value of the function $F$ at the moment of occurrence of plastic strains.

Relation (24) yields

$$\frac{t - t_0}{r} = \int\limits_{F_0}^{F} \Bigg[\phi^2(F_0) + 2\nu \int\limits_{F_0}^{x} \Big(\frac{p}{2k} - \Omega(x)\Big)\,dx\Bigg]^{-1/2} dx. \tag{27}$$

The parameter $\phi^2(F_0)$ is determined from formula (10), which implies that $F(\xi_0) = (1/(2\mu))\sin^2\alpha/\cos\alpha$:

$$\phi^2(F_0) = F'^2(\xi_0) = 2F_0(p - p_0)/(\rho g(\alpha)). \tag{28}$$

The quadrature (27) yields the relation between $F$ and $\xi - \xi_0$ for $p \geqslant p_0$.

Eliminating $F''(\xi)$ from (20) and (21) and using (22), for the plastic zone, we obtain

$$\begin{aligned}\sigma_\theta = -p + 2k\ln\frac{\sin\theta}{\sin\alpha} + \frac{1}{g(\alpha)}\ln\frac{\tan\,(\theta/2)}{\tan\,(\alpha/2)}(p - 2k\Omega(F)),\\ \sigma_\varphi = \sigma_\theta + 2k, \qquad \alpha \leqslant \theta \leqslant \theta_*\end{aligned} \tag{29}$$

and for the elastic zone,

$$\begin{aligned}\sigma_\theta = -2G\omega_0(\theta)F(\xi) - (g(\theta)/g(\alpha))(p - 2k\Omega(F)),\\ \sigma_\varphi = \sigma_\theta + 4G(\cos\theta/\sin^2\theta)F(\xi), \qquad \theta_* \leqslant \theta \leqslant \beta.\end{aligned} \tag{30}$$

1.3. *Limiting Plastic State.* We determine the minimum value of the pressure $p_*$ and the corresponding value of $t_*$ for which the elastic zone disappears, i.e., a purely plastic state occurs.

Setting $F'(\xi_*) = \phi(F_*) = 0$, where $\xi_* = t_*/r$, in (26), we obtain

$$p_* = \frac{k}{F_*}\left(\mu F_0^2 \omega_0^2(\alpha) + 2\int\limits_{F_0}^{F_*} \Omega(x)\,dx\right), \qquad F_* = F(\xi_*) = \frac{1}{2\mu}\,\frac{\sin^2\beta}{\cos\beta},$$

i.e.,

$$p_* = \frac{k}{2}\,\frac{\cos\beta}{\sin^2\beta}\left\{\left[1 - \frac{\sin^2\alpha}{\cos\alpha}\left(\frac{\cos\beta}{\sin^2\beta} + \ln\frac{\tan\,(\beta/2)}{\tan\,(\alpha/2)}\right)\right]\frac{\sin^2\alpha}{\cos\alpha} + 8\mu\int\limits_{F_0}^{F_*}\Omega(x)\,dx\right\}. \tag{31}$$

In the corresponding static problem for a conical tube, the limiting pressure is defined by the formula $p_{s*} = 2k\ln(\sin\beta/\sin\alpha)$, which was first obtained by Ivlev [3] and then by Sokolovskii [4] as the limiting case of the elastoplastic problem.

Passing to the limit $\theta \to 0$, $r \to \infty$ with fixed $y = r\theta$ and assuming that $2\mu F_* = \beta^2$ and $2\mu F_0 = \alpha^2$, from Eq. (23) we obtain $\Omega(F) = 1/2 - \mu F/\delta^2 - \ln\alpha + \ln\sqrt{2\mu F}$. Evaluating the integral in (31) and letting $\alpha r \to a$ and $\beta r \to b$, we arrive at the expression for the limiting pressure in a cylindrical tube $p_* = 2k\ln(b/a) - (k/2)(1 - a^2/b^2)$, which coincides with formula (3).

If $p > p_*$, purely plastic expansion of a conical shell occurs.

**2. Motion of the Tube with Its Inner Surface Subjected to Velocity.** Let us consider the case where the pressure is absent ($p = 0$) and at the moment $t = 0$, the inner surface of the tube is given the velocity

$$v\Big|_{t=0} = 0, \qquad \frac{\partial v}{\partial t}\Big|_{t=0} = J \quad \text{for} \quad \theta = \alpha.$$

Here $J$ is a specified constant. Then,

$$\psi(r, 0) = 0, \qquad \frac{\partial\psi}{\partial t}\Big|_{t=0} = I, \qquad I = J\sin\alpha.$$

2.1. *Linear-Elastic State.* For $t > 0$, we introduce a function $F(\xi)$ of the form $\psi(r, t) = rF(\xi)$, where $\xi = t/r$ and $0 \leqslant \xi < \infty$. In this case, the initial conditions take the form $F(0) = 0$ and $F'(0) = I$.

For $p = 0$, the differential equation (6) becomes

$$r\,\frac{\partial^2\psi}{\partial t^2} + \frac{\omega^2}{r}\,\psi(r, t) = 0$$

or $F''(\xi) + \omega^2 F(\xi) = 0$. Solving this equation with allowance for the initial conditions, we obtain

$$F(\xi) = (I/\omega)\sin\omega\xi. \tag{32}$$

For a relatively low velocity, the state of the tube is elastic.

Let plastic strain occur on the inner surface of the tube at the moment $t = t_0$. We denote the corresponding minimum value of $I$ by $I_0$. In this case, we have

$$F_0 = F(\xi_0) = \sin^2\alpha/(2\mu\cos\alpha), \tag{33}$$

and, hence,

$$\sin\omega\xi_0 = \omega\sin^2\alpha/(2\mu I\cos\alpha).$$

Assuming that $\sin\omega\xi_0 = 1$, we obtain

$$I_0 = \frac{k\omega_0(\alpha)}{\sqrt{2G\rho g(\alpha)}}\,\frac{\sin^2\alpha}{\cos\alpha}, \qquad t_0 = \frac{\pi r}{2}\sqrt{\frac{\rho g(\alpha)}{2G}}\,\frac{1}{\omega_0(\alpha)}.$$

For these values of $I_0$ and $t_0$, we have $F'(\xi_*) = 0$ and $\tau_{r\theta} = 0$.

2.2. *Elastoplastic State.* If the applied velocity $J$ is higher than $I_0/\sin\alpha$ (for $I > I_0$), the plastic zone propagates. The stress components are determined from (29) and (30) by setting $p = 0$: for the plastic zone,

$$\sigma_\theta = 2k\ln\frac{\sin\theta}{\sin\alpha} - \frac{2k}{g(\alpha)}\,\ln\frac{\tan\,(\theta/2)}{\tan\,(\alpha/2)}\,\Omega(F) \qquad (\alpha \leqslant \theta \leqslant \theta_*) \tag{34}$$

and for the elastic zone,

$$\sigma_\theta = -2G\omega_0(\theta)F(\xi) + 2k(g(\theta)/g(\alpha))\Omega(F) \qquad (\theta_* \leqslant \theta \leqslant \beta). \tag{35}$$

The expression for $\sigma_\varphi$ is the same as in Sec. 1.

The function $F(\xi)$ is determined from the differential equation $F'' + \nu\Omega(F) = 0$ (which follows from (22) for $p = 0$) and the initial conditions (28) and (33).

Setting $p = 0$ in (25), we obtain

$$\phi^2(F) = \phi^2(F_0) - 2\nu \int\limits_{F_0}^{F} \Omega(x)\, dx. \tag{36}$$

Hence,

$$\frac{t - t_0}{r} = \int\limits_{F_0}^{F} \left( \phi^2(F_0) - 2\nu \int\limits_{F_0}^{x} \Omega(x)\, dx \right)^{-1/2} dx.$$

Let us determine the minimum value of the pulse intensity $I_*$ for which the tube becomes entirely plastic. Setting $F'(\xi_*) = 0$, from (36) we obtain

$$\phi^2(F_0) = \frac{4k}{\rho g(\alpha)} \int\limits_{F_0}^{F_*} \Omega(x)\, dx, \qquad F_* = \frac{1}{2\mu} \frac{\sin^2\beta}{\cos\beta}. \tag{37}$$

It follows from (32) that

$$F'(\xi_0) = I_* \cos\omega\xi_0. \tag{38}$$

Substituting $\xi = \xi_0$ into (32) and eliminating $\xi_0$ from the resulting equation and relation (38), we obtain $I_* = \sqrt{F_0'^2 + \omega^2 F_0^2}$. Substitution of $F_0$ from (33) and $F_0'$ from (38) into the last formula yields the limiting value of $I$:

$$I_* = \frac{k}{\sqrt{2G\rho g(\alpha)}} \left( \frac{\sin^4\alpha}{\cos^2\alpha}\, \omega_0^2(\alpha) + 8\mu \int\limits_{F_0}^{F_*} \Omega(x)\, dx \right)^{1/2}.$$

We note that $I_*$ is the exact limiting value of $I$ for which $F'(\xi_*) = 0$, i.e., $\tau_{r\theta} = 0$.

2.3. *Inertial Expansion.* For $I > I_*$, inertial expansion of the tube occurs. Setting $p = 0$ in (21) and satisfying the boundary condition at the outer surface $\sigma_\theta = 0$ for $\theta = \beta$, we find the law of plastic expansion:

$$F''(\xi) = -2k \ln(\sin\beta/\sin\alpha)/(\rho g(\alpha)). \tag{39}$$

Substituting (39) into (21), for $p = 0$ we obtain

$$\frac{\sigma_\theta}{2k} = \ln\frac{\sin\theta}{\sin\alpha} - \frac{1}{g(\alpha)} \ln\frac{\sin\beta}{\sin\alpha} \ln\frac{\tan(\theta/2)}{\tan(\alpha/2)}, \qquad \sigma_\varphi = \sigma_\theta + 2k.$$

2.4. *Unloading.* At the moment $t = t_{**}$, let the expansion rate be equal to zero and plastic deformation be converted to elastic unloading. Then,

$$F'(\xi_{**}) = 0, \qquad \xi_{**} = t_{**}/r. \tag{40}$$

Integration of (39) using (40), yields

$$F(\xi) = F_{**} - \frac{k}{\rho} \frac{\ln(\sin\beta/\sin\alpha)}{g(\alpha)} (\xi - \xi_{**})^2.$$

Here $F_{**}$ is the value of $F$ for $t = t_{**}$.

The elastic unloading follows the law

$$\sigma_\theta - \sigma_\varphi - (\sigma_\theta^{**} - \sigma_\varphi^{**}) = 2G[\varepsilon_\theta - \varepsilon_\varphi - (\varepsilon_\theta^{**} - \varepsilon_\varphi^{**})], \tag{41}$$

where two asterisks denote the stresses and strains at the end of plastic expansion. Thus, at the beginning of unloading, we have

$$\sigma_\theta^{**} - \sigma_\varphi^{**} = -2k, \qquad \varepsilon_\theta^{**} - \varepsilon_\varphi^{**} = -2F_{**}\cos\theta/\sin^2\theta.$$

Introducing the function $\psi(r,t) = rF(\xi)$, we obtain

$$\varepsilon_\theta - \varepsilon_\varphi = -2F(\xi)\cos\theta/\sin^2\theta \qquad \text{for} \quad t > t_{**}.$$

From (41), it follows that

$$\sigma_\varphi - \sigma_\theta = 2k + 4G(F(\xi) - F_{**})\cos\theta/\sin^2\theta. \tag{42}$$

Integrating the equation of motion and using the condition on the surface $\theta = \alpha$, we find that

$$\sigma_\theta = 2k\ln\frac{\sin\theta}{\sin\alpha} + 2G\Big(\frac{\cos\alpha}{\sin^2\alpha} - \frac{\cos\theta}{\sin^2\theta} - \ln\frac{\tan\,(\theta/2)}{\tan\,(\alpha/2)}\Big)(F(\xi) - F_{**}) + \rho\ln\frac{\tan\,(\theta/2)}{\tan\,(\alpha/2)}\,F''(\xi). \tag{43}$$

The boundary condition on the outer surface $\sigma_\theta = 0$ for $\theta = \beta$ yields the following equation of motion during unloading:

$$F''(\xi) + \omega^2(F(\xi) - F_{**}) + 2k\ln(\sin\beta/\sin\alpha)/(\rho g(\alpha)) = 0.$$

Hence,

$$F(\xi) = F_{**} - 2k\ln(\sin\beta/\sin\alpha)(1 - \cos\omega(\xi - \xi_{**}))/(\rho g(\alpha)\omega^2). \tag{44}$$

Substitution of (44) into (42) and (43) yields

$$\frac{\sigma_\varphi}{2k} = \frac{\sigma_\theta}{2k} + 1 - \frac{2}{\omega_0^2(\alpha)}\,\frac{\cos\theta}{\sin^2\theta}\,\ln\frac{\sin\beta}{\sin\alpha}\Big(1 - \cos\frac{\omega}{r}(t - t_{**})\Big), \tag{45}$$

$$\frac{\sigma_\theta}{2k} = \ln\frac{\sin\theta}{\sin\alpha} - \frac{1}{g(\alpha)}\,\ln\frac{\sin\beta}{\sin\alpha}\,\ln\frac{\tan\,(\theta/2)}{\tan\,(\alpha/2)}$$

$$+ \frac{1}{\omega_0^2(\alpha)}\,\ln\frac{\sin\beta}{\sin\alpha}\Big[\Big(1 + \frac{\omega_0^2(\alpha)}{g(\alpha)}\Big)\ln\frac{\tan\,(\theta/2)}{\tan\,(\alpha/2)} - \frac{\cos\alpha}{\sin^2\alpha} + \frac{\cos\theta}{\sin^2\theta}\Big]\Big(1 - \cos\frac{\omega}{r}\,(t - t_{**})\Big). \tag{46}$$

Thus, during unloading for $t > t_{**}$, the tube performs harmonic oscillations with variable frequency $\omega/r$ in the presence of residual stress and strain. The unloading stage is completed at the moment $t = t_{***}$ when reverse plastic strain can occur, for which the following condition holds:

$$\sigma_\theta - \sigma_\varphi = 2k. \tag{47}$$

Combining (47), (42), and (44), we obtain the unloading time

$$(t_{***} - t_{**})/r = (2/\omega)\arcsin\big(\omega_0\sin\alpha/\sqrt{2\cos\alpha\ln(\sin\beta/\sin\alpha)}\,\big).$$

The stresses $\sigma_\theta$ and $\sigma_\varphi$ during unloading are determined from formulas (45) and (46). The occurrence of secondary plasticity in a conical tube with specified mechanical parameters can be revealed by numerical analysis.

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